On C-embedded subspaces of the Sorgenfrey plane
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1. Introduction

Recall that a subset A of a topological space X is called *functionally open* (*functionally closed*) *in* X if there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}((0,1])$ ($A = f^{-1}(0)$). Sets A and B are *completely separated in* X if there exists a continuous function $f : X \to [0,1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

A subspace E of a topological space X is

- *C-embedded (C^*-embedded) in* X if every (bounded) continuous function $f : E \to \mathbb{R}$ can be continuously extended on X;
- *z-embedded in* X if every functionally closed set in E is the restriction of a functionally closed set in X to E;
- *well-embedded in* X [7] if E is completely separated from any functionally closed set of X disjoint from E.

Clearly, every C-embedded subspace of X is C^*-embedded in X. The converse in not true. Indeed, if $E = \mathbb{N}$ and $X = \beta\mathbb{N}$, then E is C^*-embedded in X (see [4, 3.6.3]), but the function $f : E \to \mathbb{R}$, $f(x) = x$ for every $x \in E$, does not extend to a continuous function $f : X \to \mathbb{R}$.

A space X has *the property* ($C^* = C$) [11] if every closed C^*-embedded subset of X is C-embedded in X. The classical Tietze-Urysohn Extension Theorem says that if X is a normal space, then every closed subset of X is C^*-embedded and X has the property ($C^* = C$). Moreover, a space X is normal if and only if every its closed subset is z-embedded (see [9, Proposition 3.7]).

The following theorem was proved by Blair and Hager in [2, Corollary 3.6].

Theorem 1.1. *A subset E of a topological space X is C-embedded in X if and only if E is z-embedded and well-embedded in X.*

A space X is said to be *δ-normally separated* [10] if every closed subset of X is well-embedded in X. The class of δ-normally separated spaces includes all normal spaces and all countably compact spaces. Theorem 1.1 implies the following result.

Corollary 1.2. *Every δ-normally separated space has the property* ($C^* = C$).

According to [15] every C^*-embedded subspace of a completely regular first countable space is closed. The following problem is still open:

Problem 1.3. [12] *Does there exist a first countable completely regular space without property ($C^* = C$)?*

H. Ohta in [11] proved that the Niemytzki plane has the property ($C^* = C$) and asked does the Sorgenfrey plane $\mathbb{S}^2$ (i.e., the square of the Sorgenfrey line $\mathbb{S}$) have the property ($C^* = C$)?

In the given paper we obtain some necessary conditions on a set $E \subseteq \mathbb{S}^2$ to be C^*-embedded. We prove that every C^*-embedded subset of $\mathbb{S}^2$ is a hereditarily Baire subspace of $\mathbb{R}^2$. We also characterize C- and C^*-embedded subspaces of the anti-diagonal $\mathbb{D} = \{(x, -x) : x \in \mathbb{R}\}$ of $\mathbb{S}^2$. Namely, we prove that for a subspace $E \subseteq \mathbb{D}$ of $\mathbb{S}^2$ the following conditions are equivalent: (i) E is C-embedded in $\mathbb{S}^2$; (ii) E is C^*-embedded in $\mathbb{S}^2$; (iii) E is a countable G_δ-subspace of $\mathbb{R}^2$ and (iv) E is a countable functionally closed subspace of $\mathbb{S}^2$.

2. Every finite power of the Sorgenfrey line is a hereditarily α-favorable space

Recall the definition of the Choquet game on a topological space X between two players α and β. Player β goes first and chooses a nonempty open subset U_0 of X. Player α chooses a nonempty open subset V_1 of X such that $V_1 \subseteq U_0$. Following this player β must select another nonempty open subset $U_1 \subseteq V_1$ of X and α must select a nonempty open subset $V_2 \subseteq U_1$. Acting in this way, the players α and β obtain sequences of nonempty open sets $(U_n)_{n=0}^\infty$ and $(V_n)_{n=1}^\infty$ such that $U_{n-1} \subseteq V_n \subseteq U_n$ for every $n \in \mathbb{N}$. The player α wins if $\bigcap\limits_{n=1}^\infty V_n \neq \emptyset$. Otherwise, the player β wins. If there exists a rule (a strategy) such that α wins if he plays according to this rule, then X is called *α-favorable*. Respectively, X is called *β-unfavorable* if the player β has no winning strategy. Clearly, every α-favorable space X is β-unfavorable. Moreover, it is known [13] that a topological space X is Baire if and only if it is β-unfavorable in the Choquet game.

If A is a subspace of a topological space X, then $\overline{A}$ and $\mathrm{int}A$ mean the closure and the interior of A in X, respectively.

Lemma 2.1. *Let $X = \bigcup\limits_{k=1}^{n} X_k$, where X_k is an α-favorable subspace of X for every $k = 1, \dots, n$. Then X is an α-favorable space.*

Proof. We prove the lemma for $n = 2$. Let $G = G_1 \cup G_2$, where $G_i = \mathrm{int}\overline{X_i}$, $i = 1, 2$. We notice that for every $i = 1, 2$ the space $\overline{X_i}$ is α-favorable, since it contains dense α-favorable subspace. Then G_i is α-favorable as an open subspace of the α-favorable space X_i. It is easy to see that the union G of two open α-favorable subspaces is an α-favorable space. Therefore, X is α-favorable, since G is dense in X. □

Let $p = (x, y) \in \mathbb{R}^2$ and $\varepsilon > 0$. We write

$$B[p;\varepsilon) = [x, x+\varepsilon) \times [y, y+\varepsilon),$$
$$B(p;\varepsilon) = (x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon).$$

If $A \subseteq \mathbb{S}^2$ then the symbol $\mathrm{cl}_{\mathbb{S}^2} A$ ($\mathrm{cl}_{\mathbb{R}^2} A$) means the closure of A in the space $\mathbb{S}^2$ ($\mathbb{R}^2$).

We say that a space X is *hereditarily α-favorable* if every its closed subspace is α-favorable.

Theorem 2.2. *For every $n \in \mathbb{N}$ the space $\mathbb{S}^n$ is hereditarily α-favorable.*

Proof. Let $n = 1$ and $\emptyset \neq F \subseteq \mathbb{S}$. Assume that β chose a nonempty open in F set $U_0 = [a_0, b_0) \cap F$, $a_0 \in F$. If U_0 has an isolated point x in $\mathbb{S}$, then α chooses $V_1 = \{x\}$ and wins. Otherwise, α put $V_1 = [a_0, c_0) \cap F$, where $c_0 \in (a_0, b_0) \cap F$ and $c_0 - a_0 < 1$. Now let $U_1 = [a_1, b_1) \cap F \subseteq V_1$ be the second turn of β such that $a_1 \in F$ and the set $(a_1, b_1) \cap F$ has no isolated points in $\mathbb{S}$. Then there exists $c_1 \in (a_1, b_1) \cap F$ such that $c_1 - a_1 < \frac{1}{2}$. Let $V_2 = [a_1, c_1) \cap F$. Repeating this process, we obtain sequences $(U_m)_{m=0}^{\infty}$, $(V_m)_{m=1}^{\infty}$ of open subsets of F and sequences of points $(a_m)_{m=0}^{\infty}$, $(b_m)_{m=0}^{\infty}$ and $(c_m)_{m=1}^{\infty}$ such that $[a_m, b_m) \supseteq [a_m, c_m) \supseteq [a_{m+1}, b_{m+1})$, $c_m - a_m < \frac{1}{m+1}$, $c_m \in F$, $U_m = [a_m, b_m) \cap F$ and $V_{m+1} = [a_m, c_m) \cap F$ for every $m = 0, 1, \dots$. According to the Nested Interval Theorem, the sequence $(c_m)_{m=1}^{\infty}$ is convergent in $\mathbb{S}$ to a point $x^* \in \bigcap\limits_{m=0}^{\infty} V_m$. Since F is closed in $\mathbb{S}$, $x^* \in F$. Hence, $F \cap \bigcap\limits_{m=0}^{\infty} V_m \neq \emptyset$. Consequently, F is α-favorable.

Suppose that the theorem is true for all $1 \leq k \leq n$ and prove it for $k = n+1$.

Consider a set $\emptyset \neq F \subseteq \mathbb{S}^{n+1}$. Let the player β chooses a set $U_0 = F \cap \prod\limits_{k=1}^{n+1} [a_{0,k}, b_{0,k})$ with $a_0 = (a_{0,k})_{k=1}^{n+1} \in F$. Denote $U_0^+ = \prod\limits_{k=1}^{n+1} (a_{0,k}, b_{0,k})$ and consider the case $U_0^+ \cap F = \emptyset$. For every $k = 1, \dots, n+1$ we set $U_{0,k} = \{a_{0,k}\} \times \prod\limits_{i \neq k} [a_{0,i}, b_{0,i})$ and $F_{0,k} = F \cap U_{0,k}$. Since $U_{0,k}$ is homeomorphic to $\mathbb{S}^n$, by the inductive assumption the space $F_{0,k}$ is α-favorable for every $k = 1, \dots, n+1$. Then F is α-favorable according to Lemma 2.1. Now let $U_0^+ \cap F \neq \emptyset$. If there exists an isolated in $\mathbb{S}^{n+1}$ point $x \in U_0$, then α put $V_1 = \{x\}$ and wins. Assume U_0 has no isolated points in $\mathbb{S}^{n+1}$. Then there is $c_0 = (c_{0,k})_{k=1}^{n+1} \in U_0^+ \cap F$ such that $\mathrm{diam}(\prod\limits_{k=1}^{n+1} [a_{0,k}, c_{0,k})) < 1$. We put $V_1 = F \cap \prod\limits_{k=1}^{n+1} [a_{0,k}, c_{0,k})$. Let $U_1 = F \cap \prod\limits_{k=1}^{n+1} [a_{1,k}, b_{1,k})$ be the second turn of β such that $a_1 = (a_{1,k})_{k=1}^{n+1} \in F$ and $U_1 \subseteq V_1$. Again, if $U_1^+ \cap F = \emptyset$, where $U_1^+ = \prod\limits_{k=1}^{n+1} (a_{1,k}, b_{1,k})$, then, using the inductive assumption, we obtain that for every $k = 1, \dots, n+1$ the space $F \cap \big(\{a_{1,k}\} \times \prod\limits_{i \neq k} [a_{1,i}, b_{1,i})\big)$ is α-favorable. Then α has a winning strategy in F by Lemma 2.1. If $U_1^+ \cap F \neq \emptyset$ and U_1 has no isolated points in $\mathbb{S}^{n+1}$, the player α chooses a point $c_1 = (c_{1,k})_{k=1}^{n+1} \in U_1^+ \cap F$ such that $\mathrm{diam}(\prod\limits_{k=1}^{n+1} [a_{1,k}, c_{1,k})) < 1/2$ and put $V_2 = F \cap \prod\limits_{k=1}^{n+1} [a_{1,k}, c_{1,k})$. Repeating this process, we obtain sequences of points $(a_m)_{m=0}^{\infty}$, $(b_m)_{m=0}^{\infty}$ and $(c_m)_{m=0}^{\infty}$, and of sets $(U_m)_{m=0}^{\infty}$ and $(V_m)_{m=1}^{\infty}$, which satisfy the following properties:

1) $U_m = F \cap \prod\limits_{k=1}^{n+1} [a_{m,k}, b_{m,k})$;
2) $a_m \in F$, $c_m \in U_m^+ \cap F$;
3) $V_{m+1} = F \cap \prod\limits_{k=1}^{n+1} [a_{m,k}, c_{m,k})$;
4) $V_{m+1} \subseteq U_m \subseteq V_m$;
5) $\mathrm{diam}(V_{m+1}) < \frac{1}{m+1}$

for every $m = 0, 1, \ldots$. We observe that the sequence $(c_m)_{m=0}^{\infty}$ is convergent in $\mathbb{R}^{n+1}$ and $x^* = \lim\limits_{m\to\infty} c_m \in \bigcap\limits_{m=0}^{\infty} \overline{V_m} = \bigcap\limits_{m=0}^{\infty} V_m$. Since $c_m \to x^*$ in $\mathbb{S}^{n+1}$, $c_m \in F$ and F is closed in $\mathbb{S}^{n+1}$, $x^* \in F \cap \big(\bigcap\limits_{m=0}^{\infty} V_m\big)$. Hence, F is α-favorable. □

3. Every C^*-embedded subspace of $\mathbb{S}^2$ is a hereditarily Baire subspace of $\mathbb{R}^2$.

Lemma 3.1. *A set $E \subseteq \mathbb{R}^2$ is functionally closed in $\mathbb{S}^2$ if and only if*

1) *E is G_δ in $\mathbb{R}^2$; and*
2) *if F is $\mathbb{R}^2$-closed set disjoint from E, then F and E are completely separated in $\mathbb{S}^2$.*

Proof. Necessity. Let $f : \mathbb{S}^2 \to \mathbb{R}$ be a continuous function such that $E = f^{-1}(0)$. According to [1, Theorem 2.1], f is a Baire-one function on $\mathbb{R}^2$. Consequently, E is a G_δ subset of $\mathbb{R}^2$.

Condition (2) follows from the fact that every $\mathbb{R}^2$-closed set is, evidently, a functionally closed subset of $\mathbb{S}^2$.

Sufficiency. Since E is G_δ in $\mathbb{R}^2$, there exists a sequence of $\mathbb{R}^2$-closed sets F_n such that $X \setminus E = \bigcup\limits_{n=1}^{\infty} F_n$. Clearly, $E \cap F_n = \emptyset$. Then condition (2) implies that for every $n \in \mathbb{N}$ there exists a continuous function $f_n : \mathbb{S}^2 \to \mathbb{R}$ such that $E \subseteq f_n^{-1}(0)$ i $F_n \subseteq f^{-1}(1)$. Then $E = \bigcap\limits_{n=1}^{\infty} f_n^{-1}(0)$. Hence, E is functionally closed in $\mathbb{S}^2$. □

Lemma 3.2. *Let X be a metrizable space, $A \subseteq X$ be a set without isolated points and let $B \subseteq X$ be a countable set such that $A \cap B = \emptyset$. Then there exists a set $C \subseteq A$ without isolated points such that $\overline{C} \cap B = \emptyset$.*

Proof. Let d be a metric on X, which generates its topological structure. For $x_0 \in X$ and $r > 0$ we denote $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ and $B[x_0, r] = \{x \in X : d(x, x_0) \le r\}$. Let $B = \{b_n : n \in \mathbb{N}\}$. We put $A_0 = \emptyset$ and construct sequences $(A_n)_{n=1}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ of nonempty finite sets $A_n \subseteq A$ and open neighborhoods V_n of b_n which for every $n \in \mathbb{N}$ satisfy the following conditions:

$$A_{n-1} \subseteq A_n; \tag{1}$$

$$\forall x \in A_n \ \exists y \in A_n \setminus \{x\} \text{ with } d(x, y) \le \frac{1}{n}; \tag{2}$$

$$d(A_n, \bigcup_{1\le i\le n} V_i) > 0. \tag{3}$$

Let $A_1 = \{x_1, y_1\}$, where $d(x_1, y_1) \le 1$ and $x_1 \ne y_1$. We take $\varepsilon > 0$ such that $A_1 \cap B[b_1, \varepsilon] = \emptyset$ and put $V_1 = B(b_1, \varepsilon)$. Assume that we have already defined finite sets $A_1, \ldots A_k$ and neighborhoods $V_1, \ldots, V_k$ of $b_1, \ldots, b_k$, respectively, which satisfy conditions (1)–(3) for every $n = 1, \ldots, k$. Let $A_k = \{a_1, \ldots, a_m\}$, $m \in \mathbb{N}$. Taking into account that the set $D = A \setminus \bigcup\limits_{1\le i\le k} \overline{V}_i$ has no isolated points, for every $i = 1, \ldots, m$ we take $c_i \in D$ with $c_i \ne a_i$ and $d(a_i, c_i) \le \frac{1}{k+1}$. Put $A_{k+1} = A_k \cup \{c_1, \ldots, c_m\}$. Take $\delta > 0$ such that $A_{k+1} \cap B[b_{k+1}, \delta] = \emptyset$. Let $V_{k+1} = B(b_{k+1}, \delta)$. Repeating this process, we obtain needed sequences $(A_n)_{n=1}^{\infty}$ and $(V_n)_{n=1}^{\infty}$.

It remains to put $C = \bigcup\limits_{n=1}^{\infty} A_n$. □

The following results will be useful.

Theorem 3.3 ([5])**.** *A subspace E of a topological space X is C^*-embedded in X if and only if every two disjoint functionally closed subsets of E are completely separated in X.*

Theorem 3.4 ([16])**.** *The Sorgenfrey plane $\mathbb{S}^2$ is strongly zero-dimensional, i.e., for any completely separated sets A and B in $\mathbb{S}^2$ there exists a clopen set $U \subseteq \mathbb{S}^2$ such that $A \subseteq U \subseteq \mathbb{S}^2 \setminus B$.*

Recall that a space X is *hereditarily Baire* if every its closed subspace is Baire.

Theorem 3.5. *Let E be a C^*-embedded subspace of $\mathbb{S}^2$. Then E is a hereditarily Baire subspace of $\mathbb{R}^2$.*

Proof. Assume that E is not $\mathbb{R}^2$-hereditarily Baire space and take an $\mathbb{R}^2$-closed countable subspace E_0 without $\mathbb{R}^2$-isolated point (see [3]). Notice that E is $\mathbb{S}^2$-closed according to [15, Corollary 2.3]. Therefore, E_0 is $\mathbb{S}^2$-closed set. By Theorem 2.2 the space E_0 is α-favorable, and, consequently, E_0 is a Baire subspace of $\mathbb{S}^2$.

Let E_0' be a set of all $\mathbb{S}^2$-nonisolated points of E_0. Since E_0' is the set of the first category in $\mathbb{S}^2$-Baire space E_0, the set $G = E_0 \setminus E_0'$ is $\mathbb{S}^2$-dense open discrete subspace of E_0. We notice that G is $\mathbb{R}^2$-dense subspace of

E_0. By Lemma 3.2 there exists a set $C \subseteq G$ without $\mathbb{R}^2$-isolated point such that $\mathrm{cl}_{\mathbb{R}^2} C \cap E_0' = \emptyset$. We put $F = \mathrm{cl}_{\mathbb{R}^2} C \cap E_0$.

Let A and B be any $\mathbb{R}^2$-dense in F disjoint sets such that $F = A \cup B$. Evidently A and B are clopen subsets of F, since F is $\mathbb{S}^2$-discrete space. Notice that F is z-embedded in E, because F is countable. Moreover, F is $\mathbb{R}^2$-closed in E. Hence, F is $\mathbb{S}^2$-functionally closed in E. By Theorem 1.1 the set F is C-embedded in C^*-embedded in $\mathbb{S}^2$ set E. Consequently, F is C^*-embedded in $\mathbb{S}^2$. Therefore, Theorem 3.3 and Theorem 3.4 imply that there exist disjoint clopen set $U, V \subseteq \mathbb{S}^2$ such that $A = U \cap F$ and $B = V \cap F$. According to Lemma 3.1 the sets U and V are G_δ in $\mathbb{R}^2$. Let $D = \mathrm{cl}_{\mathbb{R}^2} F$. Then $U \cap D$ and $V \cap D$ are $\mathbb{R}^2$-dense in D disjoint G_δ-sets, which contradicts to the baireness of D. □

4. Every discrete C^*-embedded subspace of $\mathbb{S}^2$ is a countable G_δ-subspace of $\mathbb{R}^2$.

Lemma 4.1. *Let X be a metrizable separable space and $A \subseteq X$ be an uncountable set. Then there exists a set $Q \subseteq A$ which is homeomorphic to the set $\mathbb{Q}$ of all rational numbers.*

Proof. Let A_0 be the set of all points of A which are not condensation points A (a point $a \in X$ is called *a condensation point of A in X* if every neighborhood of a contains uncountably many elements of A). Notice that A_0 is countable, since X has a countable base. Put $B = A \setminus A_0$. Then the inequality $|A| > \aleph_0$ implies that every point of B is a condensation point of B. Take a countable subset $Q \subseteq B$ which is dense in B. Clearly, every point of Q is not isolated. Hence, Q is homeomorphic to $\mathbb{Q}$ by the Sierpiński Theorem [14]. □

Lemma 4.2. *Let E be an $\mathbb{R}^2$-hereditarily Baire z-embedded subspace of $\mathbb{S}^2$. Then the set E^0 of all isolated points of E is at most countable.*

Proof. Assume E^0 is uncountable. Notice that E^0 is an F_σ-subset of E, since E^0 is an open subset of E and $\mathbb{S}^2$ is a perfect space by [6]. Then $E^0 = \bigcup\limits_{n=1}^{\infty} E_n$, where every set E_n is closed in E. Take $N \in \mathbb{N}$ such that E_N is uncountable. According to Lemma 4.1 there exists a set $Q \subseteq E_N$ which is homeomorphic to $\mathbb{Q}$. Since Q is clopen in E_N and E_N is a clopen subset of a z-embedded in $\mathbb{S}^2$ set E, there exists a functionally closed subset Q_1 of $\mathbb{S}^2$ such that $Q = E \cap Q_1$. By Lemma 3.1 the set Q_1 is a G_δ-set in $\mathbb{R}^2$. Then Q is a G_δ-subset of a hereditarily Baire space E. Hence, Q is a Baire space, a contradiction. □

Theorem 4.3. *If E is a discrete C^*-embedded subspace of $\mathbb{S}^2$, then E is a countable G_δ-subspace of $\mathbb{R}^2$.*

Proof. Theorem 3.5 and Lemma 4.2 imply that E is a countable hereditarily Baire subspace of $\mathbb{R}^2$. According to [8, Proposition 12] the set E is G_δ in $\mathbb{R}^2$. □

The converse implication in Theorem 4.3 is not valid as Theorem 4.5 shows.

Lemma 4.4. *Let A be an $\mathbb{S}^2$-closed set, $\varepsilon > 0$ and $L(A;\varepsilon) = \{p \in \mathbb{S}^2 : B[p;\varepsilon] \subseteq A\}$. Then $L(A;\varepsilon)$ is $\mathbb{R}^2$-closed.*

Proof. We take $p_0 = (x_0, y_0) \in \mathrm{cl}_{\mathbb{R}^2} L(A;\varepsilon)$ and show that $p_0 \in L(A;\varepsilon)$. We consider $U = \mathrm{int}_{\mathbb{R}^2} B[p_0;\varepsilon]$ and prove that $U \subseteq A$. Take $p = (x, y) \in U$ and put $\delta = \min\{(x - x_0)/2, (y - y_0)/2, (x_0 + \varepsilon - x)/2, (y_0 + \varepsilon - y)/2\}$. Let $p_1 \in B(p_0;\delta) \cap L(A;\varepsilon)$. It is easy to see that $p \in B[p_1;\varepsilon)$. Then $p \in A$, since $p_1 \in L(A;\varepsilon)$. Hence, $U \subseteq A$. Then $B[p_0;\varepsilon) = \mathrm{cl}_{\mathbb{S}^2} U \subseteq \mathrm{cl}_{\mathbb{S}^2} A = A$, which implies that $p_0 \in L(A;\varepsilon)$. Therefore, $L(A;\varepsilon)$ is closed in $\mathbb{R}^2$. □

Theorem 4.5. *There exists an $\mathbb{S}^2$-closed countable discrete G_δ-subspace E of $\mathbb{R}^2$ which is not C^*-embedded in $\mathbb{S}^2$.*

Proof. Let C be the standard Cantor set on $[0,1]$ and let $(I_n)_{n=1}^{\infty}$ be a sequence of all complementary intervals $I_n = (a_n, b_n)$ to C such that $\mathrm{diam}\,(I_{n+1}) \leq \mathrm{diam}\,(I_n)$ for every $n \geq 1$. We put $p_n = (b_n; 1 - a_n)$, $E = \{p_n : n \in \mathbb{N}\}$ and $F = \{(x, 1 - x) : x \in \mathbb{R}\} \cap (C \times [0,1])$. Notice that E is a closed subset of $\mathbb{S}^2$, F is functionally closed in $\mathbb{S}^2$ and $E \cap F = \emptyset$.

Let $N' \subseteq \mathbb{N}$ be a set such that $\{b_n : n \in N'\}$ and $\{b_n : n \in \mathbb{N} \setminus N'\}$ are dense subsets of C. To show that E is not C^*-embedded in $\mathbb{S}^2$ we verify that disjoint clopen subsets

$$E_1 = \{p_n : n \in N'\} \quad \text{and} \quad E_2 = \{p_n : n \in \mathbb{N} \setminus N'\}$$

of E can not be separated by disjoint clopen subsets in $\mathbb{S}^2$. Assume the contrary and take disjoint clopen subsets W_1 and W_2 of $\mathbb{S}^2$ such that $W_i \cap E = E_i$ for $i = 1, 2$.

We prove that $W_1 \cap F$ is $\mathbb{R}^2$-dense in F. To obtain a contradiction we take an $\mathbb{R}^2$-open set O such that $O \cap F \cap W_1 = \emptyset$. Since the set $U = \mathbb{S}^2 \setminus W_1$ is clopen, $U = \bigcup\limits_{n=1}^{\infty} L(U; \frac{1}{n})$, where $L(U; \frac{1}{n}) = \{p \in \mathbb{S}^2 : B[p; 1/n) \subseteq U\}$ and the set $F_n = L(U; \frac{1}{n})$ is $\mathbb{R}^2$-closed by Lemma 4.4 for every $n \in \mathbb{N}$. Since $O \cap F$ is a Baire subspace of $\mathbb{R}^2$,

there exist $N \in \mathbb{N}$ and an $\mathbb{R}^2$-open in F subset $I \subseteq F$ such that $I \cap O \subseteq F_N \cap F \subseteq \mathbb{S}^2 \setminus E_1$. Taking into account that diam$(I_n) \to 0$, we choose $n_1 > N$ such that $b_n - a_n < \frac{1}{2N}$ for all $n \geq n_1$. Since the set $\{a_n : n \in N'\}$ is dense in C, there exists $n_2 \in N'$ such that $n_2 > n_1$ and $p = (a_{n_2}; 1 - a_{n_2}) \in I$. Clearly, $p \in F$. Consequently, $B[p; \frac{1}{N}) \cap E_1 = \emptyset$. But $p_{n_2} \in B[p, \frac{1}{N}) \cap E_1$, a contradiction.

Similarly we can show that $W_2 \cap F$ is also $\mathbb{R}^2$-dense in F.

Notice that W_1 and W_2 are G_δ in $\mathbb{R}^2$ by Lemma 3.1. Hence, $W_1 \cap F$ and $W_2 \cap F$ are disjoint dense G_δ-subsets of a Baire space F, which implies a contradiction. Therefore, E is not C^*-embedded in $\mathbb{S}^2$. □

5. A characterization of C-embedded subsets of the anti-diagonal of $\mathbb{S}^2$.

By $\mathbb{D}$ we denote the *anti-diagonal* $\{(x, -x) : x \in \mathbb{R}\}$ of the Sorgenfrey plane. Notice that $\mathbb{D}$ is a closed discrete subspace of $\mathbb{S}^2$.

Theorem 5.1. *For a set $E \subseteq \mathbb{D}$ the following conditions are equivalent:*

1) *E is C-embedded in $\mathbb{S}^2$;*
2) *E is C^*-embedded in $\mathbb{S}^2$;*
3) *E is a countable G_δ-subspace of $\mathbb{R}^2$;*
4) *E is a countable functionally closed subspace of $\mathbb{S}^2$.*

Proof. The implication (1) ⇒ (2) is obvious. The implication (2) ⇒ (3) follows from Theorem 4.3.

We prove (3) ⇒ (4). To do this we verify condition (2) from Lemma 3.1. Let F be an $\mathbb{R}^2$-closed set disjoint from E. Denote $D = F \cap \mathbb{D}$ and $U = \bigcup\limits_{p \in D} B[p; 1)$. We show that U is clopen in $\mathbb{S}^2$. Clearly, U is open in $\mathbb{S}^2$. Take a point $p_0 \in \mathrm{cl}_{\mathbb{S}^2} U$ and show that $p_0 \in U$. Choose a sequence $p_n \in U$ such that $p_n \to p_0$ in $\mathbb{S}^2$. For every n there exists $q_n \in D$ such that $p_n \in B[q_n, 1)$. Notice that the sequence $(q_n)_{n=1}^\infty$ is bounded in $\mathbb{R}^2$ and take a convergent in $\mathbb{R}^2$ subsequence $(q_{n_k})_{k=1}^\infty$ of $(q_n)_{n=1}^\infty$. Since D is $\mathbb{R}^2$-closed, $q_0 = \lim\limits_{k\to\infty} q_{n_k} \in D$. Then $p_0 \in \mathrm{cl}_{\mathbb{R}^2} B[q_0, 1)$. If $p_0 \in B[q_0, 1)$, then $p_0 \in U$. Assume $p_0 \notin B[q_0, 1)$ and let $q_0 = (x_0, y_0)$. Without loss of generality we may suppose that $p_0 \in [x_0, x_0 + 1] \times \{y_0 + 1\}$. Since $p_{n_k} \to p_0$ in $\mathbb{S}^2$, $q_{n_k} \in (-\infty, x_0] \times [y_0, +\infty)$ for all $k \geq k_0$ and $p_0 \in [x_0, x_0 + 1) \times \{y_0 + 1\}$. Then $p_0 \in \bigcup_{k=1}^\infty B[q_{n_k}, 1) \subseteq U$. Hence, U is clopen and $D = U \cap \mathbb{D}$. Since $\mathbb{D}$ and $F \setminus U$ are disjoint functionally closed subsets of $\mathbb{S}^2$, there exists a clopen set V such that $\mathbb{D} \cap V = \emptyset$ and $F \setminus U \subseteq V$. Then $F \subseteq U \cup V \subseteq \mathbb{S}^2 \setminus E$. Consequently, F and E are completely separated in $\mathbb{S}^2$. Therefore, E is functionally closed in $\mathbb{S}^2$ by Lemma 3.1.

(4) ⇒ (1). Notice that E satisfy the conditions of Theorem 1.1. Indeed, E is z-embedded in $\mathbb{S}^2$, since $|E| \leq \aleph_0$. Moreover, E is well-embedded in $\mathbb{S}^2$, since E is functionally closed. □

Remark 5.2. Notice that a subset E of $\mathbb{R}^2$ is countable G_δ if and only if it is scattered in $\mathbb{R}^2$. Indeed, assume that E is countable G_δ-set which contains a set Q without isolated points. Then Q is a G_δ-subset of $\mathbb{R}^2$ which is homeomorphic to $\mathbb{Q}$, a contradiction. On the other hand, if E is scattered, then Lemma 4.1 implies that E is countable. Since E is hereditarily Baire and countable, E is G_δ in $\mathbb{R}^2$.

Finally, we show that the Sorgenfrey plane is not a δ-normally separated space. Let $E = \{(x, -x) : x \in \mathbb{Q}\}$ and $F = \mathbb{D} \setminus E$. Then E is closed and F is functionally closed in $\mathbb{S}^2$, since F is the difference of the functionally closed set $\mathbb{D}$ and the functionally open set $\bigcup\limits_{p \in E} B[p, 1)$. But E and F can not be separated by disjoint clopen sets in $\mathbb{S}^2$, because E is not G_δ-subset of $\mathbb{D}$ in $\mathbb{R}^2$.

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